Collective mode description for a clean Fermi gas with repulsion in arbitrary dimensions:

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Collective mode description for a clean Fermi gas with repulsion in arbitrary dimensions: interaction of spin excitations and its consequences

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Abstract
We discuss a recent bosonization method developed to study clean Fermi gases
with repulsion in any dimensions. The method enables one to consider both
density and spin excitations. It is demonstrated that due to a non-Abelian
structure of the effective theory, the spin excitations interact with each other,
which leads to new logarithmic in temperature corrections to physical quantities.
Using a renormalization group scheme constructed for the effective low energy
field theory these logarithms are summed up in all orders. Temperature
dependent corrections to the specific heat and spin susceptibility are obtained
for all dimensions $d = 1, 2, 3$.

1. Introduction

The Landau theory of the Fermi liquid (FL) [1] is by now one of the most established
theories describing a system of interacting fermions. The main statement of this theory
is that the low energy behaviour of the system is similar to that for the ideal Fermi gas.
This gives a possibility to discuss experimental systems omitting the interactions and using
phenomenological constants for the effective mass of the particles, density of states at the Fermi
surface and other physical quantities.

A phenomenological description of the FL developed in the first works [1] was supported
by a diagrammatic analysis [2], which was a very good confirmation. However, the microscopic
Landau theory of [2] is based on a very strong assumption that one can single out a singular
particle–hole channel and sum proper ladder diagrams. Irreducible vertices entering the ladder
diagrams should remain finite and be analytic in the limit of small momenta and frequencies.
It is generally believed that for a Fermi gas with repulsion this assumption is correct and such
a system should behave as the Fermi liquid in dimensions $d > 1$.

Of course, the similarity between the FL and ideal Fermi gas cannot be exact and there
should be corrections at finite temperatures, frequencies or momenta. Study of unconventional

metals like high temperature superconductors, heavy fermion materials, etc, have revealed considerable deviations of their properties from those predicted by the FL. As a result, quite a few theoretical works have appeared recently where the Landau FL theory was discussed [3–7].

The corrections to physical quantities become especially interesting when they are non-analytic functions of temperature, frequency or momentum. This means that physical quantities like $C(T)/T$, where $C(T)$ is the specific heat and T is the temperature, $\chi(T)$, where $\chi(T)$ is the spin susceptibility, etc, cannot be represented at low temperatures as a series in T^2, which contrasts with the ideal Fermi gas. Such non-analytical corrections were studied in a number of publications using diagrammatic expansions in the electron–electron interaction [8–18]. As a result of this investigation, it is well known by now that in $d = 3$ the next-to-leading term in $C(T)/T$ is proportional to $T^2 \ln T$ [8–11]. In 2D, the non-analytical corrections to $C(T)/T$ and $\chi(T, Q)$ scale as T and max$\{T, v_0 Q\}$, respectively (Q is the wavevector of the external magnetic field) [13–19].

The existence of the non-analytic corrections to the physical quantities is not accidental. In fact, all the singular corrections to the thermodynamic quantities may be understood in terms of contributions of low lying collective excitations; see, e.g., [19, 17]. At the same time, explicit calculations with the conventional diagrammatic technique are not simple already in the lowest orders of the perturbation theory. Therefore, selecting diagrams in order to group them into the collective modes is a rather difficult task.

In this paper we present a new method of calculations for a clean Fermi gas with a repulsion that enables us to 'integrate out' electron degrees of freedom at the beginning of all calculations and reduce the initial fermionic model to a model of low lying excitations. The method is based on using equations for quasiclassical Green functions and includes both the density and spin excitations. We loosely call our method *bosonization* but it differs from earlier higher dimensional bosonization schemes [20–22, 6, 23–28, 5] based on the assumption of a long range electron–electron interaction. As a result, only density excitations were considered, while the spin excitations were not included in these schemes (to be more precise, the spin excitations are not affected by the long range interaction and therefore they were not included in the scheme).

The density excitations are described by a scalar function and the effective interaction $V_c(k, \omega)$ between them vanishes in the limit $\omega, k \to 0$. This means that the interaction may lead only to a renormalization of parameters characterizing physical quantities (unless $V_c(k, \omega)$ is long ranged) and not to new effects. In other words, the bosonization of [20–22, 6, 23–28, 5] enables one to reduce a system with a long range interaction to a model of free bosons, which is very similar to writing the Tomonaga–Luttinger model for one dimensional electron systems (see e.g. [29]). This method is often referred to as bosonization.

What we want to present now is a scheme that enables one to consider for arbitrary interactions both the density and spin excitations on equal footing. The relevant variable describing the spin excitations is a 2×2 matrix and our method resembles to some extent a non-Abelian bosonization. The effective interaction between the spin excitations does not vanish in the limit $\omega, k \to 0$ and leads to new logarithmic contributions to scattering amplitudes coming from low energies of the order of T. As a result, physical quantities depend on the logarithmically renormalized amplitudes, which changes their temperature dependence. It is important to emphasize that the 'infrared' logarithmic divergences that we find in the limit $T \to 0$ exist in *any* dimensions (including $d = 1$) and have nothing in common with the 'ultraviolet' logarithmic divergence (divergence originating from short distances) discussed for 2D systems long ago [30]. The latter does not lead to any additional dependence on temperature and can be absorbed into parameters characterizing the FL.

We display in the subsequent sections the main idea of our method and new results that have been obtained recently. In section 2 we make a Hubbard–Stratonovich transformation and

derive quasiclassical equations for the density and spin excitations representing their solution in terms of an integral over supervectors. The interaction between the spin excitations gives logarithms that are summed in section 3 using a new renormalization group (RG) scheme. We calculate the specific heat and spin susceptibility in section 4 and discuss the results in section 5.

The method and the calculation of the specific heat were presented for the first time in [31]. Using this method the calculation of the spin susceptibility was carried out later in [32]. Our presentation here is based on these publications but we concentrate rather on explaining the main steps of the derivation than on explicit calculations. All necessary details can be found in [31, 32].

2. Spin and density excitations and their contribution to thermodynamics

2.1. Singling out slow pairs and Hubbard–Stratonovich transformation

In this section we show how one can reduce calculation of the partition function of the interacting fermions to calculation of the partition function of the density and spin excitations. It will be demonstrated that the density excitations are described by a model of free bosons, whereas spin excitations interact with each other.

We start the discussion writing the original partition function Z in terms of a functional integral over classical anticommuting variables $\chi_\sigma(x)$ ($x = \{\mathbf{r}, \tau\}$ and σ labels the spin)

$$Z = \int \exp(-\mathcal{S}) D\chi \, D\chi^*. \tag{1}$$

The action $\mathcal{S}$ entering equation (1) has the form

$$\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_{\text{int}}, \tag{2}$$

where the term $\mathcal{S}_0$,

$$\mathcal{S}_0 = \sum_\sigma \int \chi_\sigma^*(x) \left(-\frac{\partial}{\partial \tau} - \frac{\hat{\mathbf{p}}^2}{2m} + \epsilon_{\text{F}} \right) \chi_\sigma(x) \, \mathrm{d}x \tag{3}$$

stands for the action of free fermions (ϵ_{F} is the Fermi energy, m is the mass and $\hat{\mathbf{p}}$ is the momentum operator). Equations (2) and (3) are written in the Matsubara representation with the imaginary time τ, such that the field variables $\chi(\mathbf{r}, \tau)$ are antiperiodic in τ

$$\chi(\mathbf{r}, \tau) = -\chi(\mathbf{r}, \tau + 1/T). \tag{4}$$

The term $\mathcal{S}_{\text{int}}$ in equation (2) describes the fermion–fermion interaction,

$$\mathcal{S}_{\text{int}} = \tfrac{1}{2} \sum_{\sigma, \sigma'} \int \chi_\sigma^*(x) \chi_{\sigma'}^*(x') v(x - x') \chi_{\sigma'}(x') \chi_\sigma(x) \, \mathrm{d}x \, \mathrm{d}x', \tag{5}$$

where $v(x - x') = V(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau')$ and $V(\mathbf{r} - \mathbf{r}')$ is the potential of the interaction.

The presence of a magnetic field $\mathbf{b}$ acting on spin can be accounted for by adding an addition term $\mathcal{S}_{\text{b}}$ to the action

$$\mathcal{S}_{\text{b}} = \int \mathrm{d}x \, \chi_\sigma^*(x) \mathbf{b} \boldsymbol{\sigma}_{\sigma\sigma'} \chi_{\sigma'}(x). \tag{6}$$

Inclusion of this term is necessary for calculation of the spin susceptibility $\chi(T)$.

The functional integral over $\chi_\sigma(x)$ in equation (1) is too complicated to be calculated exactly and we restrict ourselves to the case of a weak interaction. A stronger interaction may renormalize the coupling constants but does not seem to change the temperature behaviour.

In order to reduce the fermionic model to the model of the low lying excitations we single out in the interaction term $\mathcal{S}_{int}$, equation (5), pairs $\chi^* \chi$ slowly varying in space and write the effective interaction $\mathcal{S}_{int}$ as

$$\mathcal{S}_{int} \rightarrow \frac{1}{2} \sum_{\sigma,\sigma'} \int dP_1 \, dP_2 \, dK \{ V_2 \chi_\sigma^*(P_1) \chi_\sigma(P_1 + K) \chi_{\sigma'}^*(P_2) \chi_{\sigma'}(P_2 - K)$$

$$- V_1(\mathbf{p}_1 - \mathbf{p}_2) \chi_\sigma^*(P_1) \chi_{\sigma'}(P_1 + K) \chi_{\sigma'}^*(P_2) \chi_\sigma(P_2 - K) \}. \tag{7}$$

In equation (7), $P_i = (\mathbf{p}_i, \varepsilon_{n_i})$, where $\mathbf{p}_i$ is the momentum and $\varepsilon_{n_i} = \pi T (2n_i + 1)$ are Matsubara fermionic frequencies ($i = 1, 2$). As concerns K, it has the form $K = (\mathbf{k}, \omega_n)$, where $\omega_n = 2\pi T n$ are Matsubara bosonic frequencies.

The symbols of the integration $\int dP_i$ and $\int dK$ in equation (7) read as follows

$$\int dP_i(\cdots) = T \sum_{\varepsilon_{n_i}} \frac{d^d \mathbf{p}}{(2\pi)^d}(\cdots), \qquad \int dK(\cdots) = T \sum_{\omega_n \neq 0} f(\mathbf{k}) \frac{d^d \mathbf{k}}{(2\pi)^d}(\cdots) \tag{8}$$

where

$$f(\mathbf{k}) = f_0(k r_0) \tag{9}$$

and $k = |\mathbf{k}|$. The function $f_0(t)$ has the following asymptotics: $f_0(t) = 1$ at $t = 0$ and $f(t) \rightarrow 0$ at $t \rightarrow \infty$.

The function $f(\mathbf{k})$ in equation (9) is written in order to cut off large momenta k. The parameter r_0 is the minimal length in the theory and we assume that $r_0 \gtrsim p_F^{-1}$. So, the momenta k are cut by the maximal momentum $k_c = r_0^{-1}$ and we avoid double counting when calculating the partition function Z.

Introducing the cut-off r_0 means that the pairs written in equation (7) vary slowly in space. Accordingly, we neglect the dependence of V_1 and V_2 on the momentum $\mathbf{k}$ in equation (7). Although smaller than the Fermi momentum p_F and the Fermi energy ε_F, the cut-off k_c is larger than all other momenta in the model.

Additional decoupling in the Cooper channel is not included, since this would amount to overcounting of relevant scattering processes [31]. To be brief, the most important is parallel or antiparallel motion of the particles and one needs to consider only forward and backward scattering. In this limit, adding the Cooper channel would mean double counting. This is not so for disordered systems, where all scattering angles are important and where one should take into account the Cooper channel [33].

For a short range potential we can further simplify our considerations by setting $V_2 = V(|\mathbf{q}| \ll p_F)$. Since important momenta are close to the Fermi surface we can write $V_1(\theta_{12}) = V(\mathbf{p}_1 - \mathbf{p}_2) = V(2p_0 \sin(\frac{\theta_{12}}{2}))$, where θ_{12} is the angle between momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, $\theta_{12} = \widehat{\mathbf{p}_1 \mathbf{p}_2}$.

For the further development of the theory it will be crucial to separate explicitly interactions in the triplet and singlet channel. Making the notations

$$V_s(\theta_{12}) = V_2 - \frac{1}{2} V_1(\theta_{12}), \qquad V_t(\theta_{12}) = \frac{1}{2} V_1(\theta_{12}) \tag{10}$$

one can represent the interaction term in the form of a sum of charge and spin parts,

$$\tilde{\mathcal{S}}_{int} = \mathcal{S}_{int,s} + \mathcal{S}_{int,t}, \tag{11}$$

$$\mathcal{S}_{int,s} = \frac{1}{2} \int dp_1 \, dp_2 \, dq \, \rho(p_1, -q) V_s(\theta_{12}) \rho(p_2, q), \tag{12}$$

$$\mathcal{S}_{int,t} = -\frac{1}{2} \int dp_1 \, dp_2 \, dq \, \mathbf{S}(p_1, -q) V_t(\theta_{12}) \mathbf{S}(p_2, q), \tag{13}$$

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where the charge $\rho(p, q)$ and spin densities $\mathbf{S}(p, q)$ are

$$\rho(p, q) = \chi^\dagger\left(p - \frac{q}{2}\right)\chi\left(p + \frac{q}{2}\right), \qquad \mathbf{S}(p, q) = \chi^\dagger\left(p - \frac{q}{2}\right)\boldsymbol{\sigma}\chi\left(p + \frac{q}{2}\right), \qquad (14)$$

and we have changed to a spinor notation $\chi = (\chi_\uparrow, \chi_\downarrow)$.

In order to simplify the presentation we do not write for a while the function f assuming that the variables ρ and S are not equal to zero for small q only, which corresponds to a slow variation of these variables in space.

Having written the interaction term $\tilde{\mathcal{S}}_{\text{int}}$ in the form of equation (11) we next decouple it using a Hubbard–Stratonovich transformation with a field

$$\phi_\mathbf{n}(x) \equiv i\varphi_\mathbf{n}(x) + \boldsymbol{\sigma}\mathbf{h}_\mathbf{n}(x). \qquad (15)$$

Here $\varphi_\mathbf{n}(x)$ and $\mathbf{h}_\mathbf{n}(x)$ are real bosonic fields, so that $\phi_\mathbf{n}(\mathbf{r}, \tau) = \phi_\mathbf{n}(\mathbf{r}, \tau + \beta)$ and $\mathbf{n}$ is the direction of momentum $\mathbf{p}$ on the Fermi surface, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. The result is the following representation of the partition function (we omit for a while the external field $\mathbf{b}$):

$$\mathcal{Z} = \int \mathcal{D}\phi \, W_s[\varphi] W_t[\mathbf{h}] \mathcal{Z}[\mathbf{h}, \varphi]. \qquad (16)$$

The weight functions W_s, W_t are shown below in equations (20) and (21). The partition function $\mathcal{Z}[\mathbf{h}, \varphi]$ describes the fermion motion for a fixed configuration of fields $\mathbf{h}, \varphi$

$$\mathcal{Z}[\phi] = \int D(\chi^*, \chi) \exp(-S_{\text{eff}}[\phi]) \qquad (17)$$

where the effective action $\mathcal{S}_{\text{eff}}$ has the form

$$\mathcal{S}_{\text{eff}}[\phi] = \mathcal{S}_0 + \int d\mathbf{p} \, d\mathbf{r}_1 \, d\mathbf{r}_2 \, \chi^\dagger(\mathbf{r}_1, \tau)\phi_\mathbf{n}\left(\frac{\mathbf{r}_1 + \mathbf{r}_2}{2}\right)\chi(\mathbf{r}_2, \tau)e^{i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2)}. \qquad (18)$$

Now we can write down a representation of the partition function in the presence of the magnetic field as a weighted integral over field configurations

$$\mathcal{Z} = \int \mathcal{D}\phi \, W_s[\varphi] W_t[\mathbf{h}] \mathcal{Z}[\boldsymbol{\phi}], \qquad (19)$$

where the weights $W_s[\varphi]$ and $W_t[\mathbf{h}]$ are

$$W_s[\varphi] = \exp\left[-\frac{1}{2}\int d\hat{\mathbf{n}}_1 \, d\hat{\mathbf{n}}_2 \, d\mathbf{q} \, d\tau \, \varphi_{\mathbf{n}_1}^*(\mathbf{q}, \tau) V_s^{-1}(\theta_{12})\varphi_{\mathbf{n}_2}(\mathbf{q}, \tau)\right] \qquad (20)$$

$$W_t[\mathbf{h}] = \exp\left[-\frac{1}{2}\int d\hat{\mathbf{n}}_1 \, d\hat{\mathbf{n}}_2 \, d\mathbf{q} \, d\tau \, \mathbf{h}_{\mathbf{n}_1}^\dagger(\mathbf{q}, \tau) V_t^{-1}(\theta_{12})\mathbf{h}_{\mathbf{n}_2}(\mathbf{q}, \tau)\right]. \qquad (21)$$

Equations (16)–(21) are the final result of this subsection. We see that the original problem of the electron with interaction is replaced by a problem of electron motion in a potential and a magnetic field slowly varying in space. The condition of the slow variations follows from our separation into slow pairs. As we will see, at low temperature and weak interactions the slow variations of the fields $\mathbf{h}$ and φ give the main contribution to the physical quantities.

2.2. Quasiclassical equations

What we should do is to calculate quantities for any $\mathbf{h}$ and φ and integrate these quantities over these fields. First, we introduce the Green functions $G_{\sigma,\sigma'}^\phi(x, x')$ corresponding to the action $\mathcal{S}_{\text{eff}}[\phi]$, equation (18), as follows:

$$G_{\sigma,\sigma'}^\phi(x, x') = \mathcal{Z}^{-1}[\phi] \int \chi_\sigma(x)\chi_{\sigma'}^*(x') \exp(-\mathcal{S}_{\text{eff}}[\phi]) D\chi \, D\chi^*. \qquad (22)$$

As the fluctuating field ϕ, equation (15), varies slowly in space one can derive quasiclassical equations for the function $G^\phi_{\sigma,\sigma'}(x, x')$. The method of the derivation is well known [34]. One should perform the Fourier transform with respect to the difference $\mathbf{r} - \mathbf{r}'$ and assume that the Green function slowly depends on $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$. Introducing the quasiclassical Green function $g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau')$ in the standard way [34, 35]

$$g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau') = i \int_{-\infty}^{\infty} G^\phi_\mathbf{p}(\mathbf{R}, \tau, \tau') \frac{d\xi}{\pi}, \qquad \xi = \frac{\mathbf{p}^2}{2m} - \epsilon_F \qquad (23)$$

we come to the following equation for this function:

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau'} - i v_F \mathbf{n} \nabla \right) g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau') + g^\phi_\mathbf{n}(\mathbf{R}; \tau, \tau') \phi_\mathbf{n}(\mathbf{R}, \tau') - \phi_\mathbf{n}(\mathbf{R}, \tau) g^\phi_\mathbf{n}(\mathbf{R}; \tau, \tau') = 0 \qquad (24)$$

where $\mathbf{n}^2 = 1$, such that $p_F \mathbf{n}$ is a vector on the Fermi surface.

In principle, one could derive equation (24) more accurately, which would produce additional terms containing space derivatives of the functions $\phi_\mathbf{n}(\mathbf{R}, \tau)$ and $g^\phi_\mathbf{n}(\mathbf{R}; \tau, \tau')$ in the second line. However, the additional derivatives would compensate infrared singularities we are interested in. This is the reason why we neglect them. At the same time, no higher derivatives arise in the first line in equation (24) and this term is exact.

The function $g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau')$ must obey the antiperiodicity conditions

$$g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau') = -g^\phi_\mathbf{n}(\mathbf{R}, \tau + 1/T, \tau') = -g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau' + 1/T) \qquad (25)$$

that follow from equation (4).

Equation (24) is linear and therefore is not sufficient to find $g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau')$ unambiguously. However, the same equation as equation (24) can be written for the function g^2

$$g^2(\mathbf{R}, \tau, \tau') = \int_0^{1/T} g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau'') g^\phi_\mathbf{n}(\mathbf{R}, \tau'', \tau') \, d\tau''. \qquad (26)$$

An obvious solution for g^2 can be written as

$$g^2(\mathbf{R}, \tau, \tau') = c\delta(\tau - \tau') \qquad (27)$$

where c is an arbitrary constant. It can be fixed assuming that the fermion–fermion interaction is present only in a finite, although macroscopic, part of the space. Then, outside this space we come to the Green function of a free fermion gas satisfying equation (27) with $c = 1$. So, we come to the equation

$$\int_0^{1/T} g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau'') g^\phi_\mathbf{n}(\mathbf{R}, \tau'', \tau') \, d\tau'' = \delta(\tau - \tau'). \qquad (28)$$

Equation (28) complements equation (24) and these equations are sufficient to find the function $g^\phi_\mathbf{n}(\mathbf{R}, \tau, \tau')$ for any function $\phi_\mathbf{p}(\mathbf{R}, \tau)$. After this, in order to calculate physical quantities, one should perform a proper averaging over $\phi_\mathbf{p}(\mathbf{R}, \tau)$ with the weights $W_{s,t}[\phi]$, equations (20) and (21). In the next subsection we will show how to express the partition function $\mathcal{Z}[\phi]$, equation (17), in terms of the solution of these equations but now let us reduce equations (24) and (28) to a more simple form.

Now we come to the main point of the method proposed here. We notice that the quasiclassical Green function of the free fermion gas is singular and can be written as

$$g_{0\mathbf{n}}(\tau - \tau') = -iT \, \mathrm{Re}[\sin^{-1} \pi T (\tau - \tau' - i\delta)] \qquad (29)$$

where $\delta \to +0$. Of course, the Green function $g_{0\mathbf{n}}(\tau - \tau')$ satisfies equations (24), (25) and (28).

As concerns arbitrary $\phi_{\mathbf{n}}(\mathbf{R}, \tau)$, we look for the general solution of equations (24) and (28) in the following form:

$$g_{\mathbf{n}}^{\phi}(\mathbf{R}, \tau, \tau') = T_{\mathbf{n}}(\mathbf{R}, \tau)g_0(\tau - \tau')T_{\mathbf{n}}^{-1}(\mathbf{R}, \tau') \tag{30}$$

where $T_{\mathbf{n}}(\mathbf{r}, \tau)$ is a spin matrix satisfying the condition

$$T_{\mathbf{n}}(\mathbf{R}, \tau) = T_{\mathbf{n}}(\mathbf{R}, \tau + 1/T). \tag{31}$$

The representation of the Green function in the form of equation (30) is a generalization of the Schwinger ansatz [36]. The form given by equation (30) is consistent with equation (28), and what remains to be done is to find the proper matrix $T_{\mathbf{n}}(\mathbf{R}, \tau)$, such that equation (24) is satisfied.

A straightforward manipulation [31] enables one to reduce equations (24) and (28) to the following form:

$$\left(-\frac{\partial}{\partial \tau} + \mathrm{i}v_{\mathrm{F}}\mathbf{n}\nabla_{\mathbf{R}}\right)M_{\mathbf{n}}(x) + [\phi_{\mathbf{n}}(x), M_{\mathbf{n}}(x)] = -\frac{\partial \phi_{\mathbf{n}}(x)}{\partial \tau} \tag{32}$$

where

$$M_{\mathbf{n}}(x) = \frac{\partial T_{\mathbf{n}}(x)}{\partial \tau}T_{\mathbf{n}}^{-1}(x) \tag{33}$$

and the symbol [,] stands for the commutator.

Using the representation equation (15) for the matrix $\phi_{\mathbf{n}}(x)$ and writing the matrix $M_{\mathbf{n}}(x)$ as

$$M_{\mathbf{n}}(x) = \rho_{\mathbf{n}}(x) + \mathbf{S}_{\mathbf{n}}(x)\boldsymbol{\sigma}, \tag{34}$$

where $\rho_{\mathbf{n}}(x)$ is a scalar function and $\mathbf{S}_{\mathbf{n}}(x)$ is a three-dimensional vector, we reduce equation (32) to two independent equations for $\rho_{\mathbf{n}}(x)$ and $\mathbf{S}_{\mathbf{n}}(x)$,

$$\left(-\frac{\partial}{\partial \tau} + \mathrm{i}v_{\mathrm{F}}\mathbf{n}\nabla_{\mathbf{R}}\right)\rho_{\mathbf{n}}(x) = -\mathrm{i}\frac{\partial \varphi_{\mathbf{n}}(x)}{\partial \tau} \tag{35}$$

$$\left(-\frac{\partial}{\partial \tau} + \mathrm{i}v_{\mathrm{F}}\mathbf{n}\nabla_{\mathbf{R}}\right)\mathbf{S}_{\mathbf{n}}(x) + 2\mathrm{i}[\mathbf{h}_{\mathbf{n}}(x) \times \mathbf{S}_{\mathbf{n}}(x)] = -\frac{\partial \mathbf{h}_{\mathbf{n}}(x)}{\partial \tau}. \tag{36}$$

Equations (35) and (36) are the final quasiclassical equations that will be used for further calculations. We emphasize that equations (35) and (36) are obtained from equations (24) and (28) without making any approximation. The variable $\rho_{\mathbf{n}}(x)$ describes collective density excitations, whereas the variable $\mathbf{S}_{\mathbf{n}}(x)$ stands for spin ones.

Equations (35) and (36) describing these excitations are remarkably different from each other. Equation (35) for the density is rather simple. This is what one obtains using the high dimensional bosonization of [21, 22, 6, 23–25, 27, 28, 5] from an eikonal equation. Of course, we could take into account gradients of the field $\varphi_{\mathbf{n}}(x)$ and this would lead to additional terms in the LHS of equation (35). However, this does not lead to new physical effects.

In contrast, equation (36) is highly non-trivial due to the presence of $\mathbf{h}_{\mathbf{n}}(x)$ in the LHS of the equation. Actually, the homogeneous part of equation (36) is just the equation of motion of a classical magnetic moment in the external magnetic field. We will see that the form of equation (36) will result in very non-trivial effects that will be considered later.

The presence of the second term in the LHS of equation (36) is a consequence of a non-Abelian character of the variables describing the spin excitations. In this respect our method resembles the non-Abelian bosonization well known for one-dimensional systems [29].

It is important to emphasize that the variables $\rho_{\mathbf{n}}(x)$ and $\mathbf{S}_{\mathbf{n}}(x)$ depend not only on the time and coordinate but also on the position of the vector $\mathbf{n}$, that determines the position on the Fermi surface. This dependence is usual for kinetic equations. In the previous attempts to construct a higher dimensional bosonization [21, 22, 6, 23–25, 27, 28, 5] the corresponding variable arose from a 'patching' of the Fermi surface.

2.3. Partition function

What remains to be done is to express the partition function $\mathcal{Z}[\Phi]$, equations (17) and (18), in terms of the variables $\rho_{\mathbf{n}}(x)$ and $\mathbf{h}_{\mathbf{n}}(x)$. Integrating over χ, χ^* in equation (17) and using equations (18) and (3) for $\mathcal{S}_{\mathrm{eff}}[\phi]$ we write $\mathcal{Z}[\phi]$ in the form

$$\mathcal{Z}[\phi] = \exp\left[\mathrm{Tr}\int \ln\left(-\frac{\partial}{\partial\tau} - \frac{\hat{\mathbf{p}}^2}{2m} + \epsilon_{\mathrm{F}} + \phi_{\mathbf{n}}(x)\right)\mathrm{d}x\right]. \tag{37}$$

Equation (37) can further be represented as

$$\mathcal{Z}[\phi] = Z_0\exp\left[\mathrm{Tr}\int\int_0^1 G^\phi(x, x; u)\phi_{\mathbf{n}}(x)\,\mathrm{d}x\,\mathrm{d}u\right] \tag{38}$$

where the Green function $G^\phi(\mathbf{r}, \mathbf{r}; \tau, \tau; u)$ is the solution of the equation

$$\left(-\frac{\partial}{\partial\tau} - \frac{\hat{\mathbf{p}}^2}{2m} + \epsilon_{\mathrm{F}} + u\phi_{\mathbf{n}}(x)\right)G^\phi(x, x'; u) = \delta(x - x'). \tag{39}$$

We see that the calculation of the partition function reduces to the calculation of the Green function $G^\phi(x, x'; u)$, that differs from the Green functions calculated in the preceding subsection by the replacement $\phi(x) \to u\phi(x)$, where u is a parameter in the interval $(0, 1)$. This means that calculating the Green function we can repeat all the transformations we have performed previously.

The Green function $G^\phi(\mathbf{r}, \tau; \mathbf{r}, \tau'; u)$ at coinciding points can be written in terms of the quasiclassical Green function $g_{\mathbf{n}}^\phi$ as

$$G^\phi(x, x'; u) = -\mathrm{i}\pi\nu\int g_{\mathbf{n}}^\phi(\mathbf{r}; \tau, \tau'; u)\,\mathrm{d}\mathbf{n} \tag{40}$$

where ν is the density of states on the Fermi surface (without taking into account the double degeneracy due to spin).

Next, we use the representation equation (30), and expand the function $T_{\mathbf{n}}^{-1}(\mathbf{r}, \tau')$ in $\tau' - \tau$. The contribution from $g_{0\mathbf{n}}(\tau - \tau')$, equation (29), vanishes at $\tau = \tau'$. Using equations (33) and (34) we obtain finally

$$\mathcal{Z}[\phi] = \mathcal{Z}_0\mathcal{Z}_\varphi\mathcal{Z}_{\mathbf{h}} \tag{41}$$

where

$$\mathcal{Z}_\varphi = \exp\left[-2\mathrm{i}\nu\int\int_0^1 \rho_{\mathbf{n}}(x, u)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\mathbf{n}\,\mathrm{d}u\right], \tag{42}$$

$$\mathcal{Z}_{\mathbf{h}} = \exp\left[-2\nu\int\int_0^1 \mathbf{S}_{\mathbf{n}}(x, u)\mathbf{h}_{\mathbf{n}}(x)\,\mathrm{d}x\,\mathrm{d}\mathbf{n}\,\mathrm{d}u\right]. \tag{43}$$

The functions $\rho_{\mathbf{n}}(x, u)$ and $\mathbf{S}_{\mathbf{n}}(x, u)$ should be found from the equations

$$\hat{L}_{u=0}(\mathbf{n})\rho_{\mathbf{n}}(x) = -\mathrm{i}u\frac{\partial\varphi_{\mathbf{n}}(x)}{\partial\tau}, \qquad \hat{L}_u(\mathbf{n})\mathbf{S}_{\mathbf{n}}(x) = -u\frac{\partial\mathbf{h}_{\mathbf{n}}(x)}{\partial\tau}. \tag{44}$$

In equation (44), the operator $\hat{L}_u$ has the form

$$\hat{L}_u(\mathbf{n}) = -\frac{\partial}{\partial\tau} + \mathrm{i}v_{\mathrm{F}}(\mathbf{n}\nabla_{\mathbf{r}}) + 2\mathrm{i}u\hat{h} \tag{45}$$

where the matrix $\hat{h}$ is

$$\hat{h} = \begin{pmatrix} 0 & -h_z & h_y \\ h_z & 0 & -h_x \\ -h_y & h_x & 0 \end{pmatrix} \tag{46}$$

and h_x, h_y, and h_z are the components of the vector $\mathbf{h}$ ($\hat{h}\mathbf{a} = [\mathbf{h} \times \mathbf{a}]$ for any vector $\mathbf{a}$).

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The functions $\mathbf{S_n}(x)$ and $\rho_{\mathbf{n}}(x)$ are periodic in τ

$$\mathbf{S_n}(\mathbf{r}, \tau) = \mathbf{S_n}(\mathbf{r}, \tau + 1/T). \tag{47}$$

The accuracy of equations (20) and (21) can be somewhat improved by making the substitution $V_{s,t} \to \hat{\Gamma}_{s,t}/\nu$, where $\hat{\Gamma}_{s,t}$ is the scattering amplitude for the singlet and triplet channels respectively.

Thus, we have reduced the study of the system of the interacting fermions to investigation of a system of bosonic density and spin excitations. Therefore the word 'bosonization' is most suitable for our approach. We see that the method should work in any dimension. At the same time, it is more general than the scheme of the high dimensional bosonization of [21, 22, 6, 23–25, 27, 28, 5] because we can consider the spin excitations, that are much more interesting than the density ones. The presence of the non-trivial third term in equations (36) and (45) is a consequence of the non-Abelian character of the excitations. In contrast, the previous schemes worked only for Abelian density excitations and it is simply impossible to write any interaction of ρ with an external field φ without using time or space derivatives.

3. Effective field theory and renormalization group equations

3.1. Infrared logarithmic divergences

Equations (41)–(47) and (19)–(21) are sufficient for calculating low energy contributions to the thermodynamic quantities. Calculation for the density excitations is not difficult because the first equation (44) can easily be solved explicitly. As concerns the spin excitations one may seek the solution of the second equation (44) expanding the operator L_u, equation (44), in $\hat{h}$. It turns out that in the limit $T \to 0$ terms of the perturbation theory for scattering amplitudes are logarithmically divergent in any dimension and one has to sum an infinite series.

In order to see the origin of this divergence, let us consider the expression $K(\mathbf{n}, -\mathbf{n}') = \int L_u^{-1}(\mathbf{n}; x, x')L_u^{-1}(-\mathbf{n}'; x', x)\,\mathrm{d}x'$. Using the Fourier transform in the coordinates and time we bring this expression in the limit $T \to 0$ to the form

$$K(\mathbf{n}, -\mathbf{n}') = \int \frac{\mathrm{d}\omega\, \mathrm{d}^d\mathbf{k}}{(2\pi)^{d+1}} \frac{1}{\mathrm{i}\omega - v_F \mathbf{k}\mathbf{n}} \frac{1}{\mathrm{i}\omega + v_F \mathbf{k}\mathbf{n}'}. \tag{48}$$

If $\mathbf{n}$ is parallel to $\mathbf{n}'$ we can integrate separately over the parallel $k_{\parallel}$ and perpendicular $\mathbf{k}_{\perp}$ (with respect to the vector $\mathbf{n}$) components of the vector $\mathbf{k}$. In this case the integrand does not depend on $\mathbf{k}_{\perp}$ and formally diverges at large $|\mathbf{k}_{\perp}|$. However, we assumed that the momenta cannot be very large and therefore the maximal $|\mathbf{k}_{\perp}|$ are of the order of r_0^{-1} from equation (9), which provides the convergence of the integral over $\mathbf{k}_{\perp}$.

In contrast, the integral over ω and $k_{\parallel}$ diverges at *small* values of these variables. These are infrared divergences and they lead to important contributions to the thermodynamic quantities. Estimating the value of K we come to the following expression:

$$K(\mathbf{n}, -\mathbf{n}') \sim r_0^{1-d} v_F^{-1} \ln(\max\{T v_F^{-1} r_0, |\mathbf{n} - \mathbf{n}'|\}). \tag{49}$$

Products of the function $K(\mathbf{n}, -\mathbf{n}')$ arise when calculating scattering amplitudes with the help of the perturbation theory and we see that in the limit $T \to 0$ one should sum an infinite series. This is not an easy task but the consideration simplifies if we reformulate the problem of calculation of the partition function in terms of a field theory. Then we will be able to use for summation of the logarithms a renormalization group technique.

3.2. Low energy supersymmetric field theory

Now, our task is to solve equations (44), substitute the solution into equations (41), (43) and, using the obtained expression for $\mathcal{Z}[\phi]$, calculate the integral over the fields $\varphi_\mathbf{n}$ and $\mathbf{h_n}$. This procedure is somewhat similar to what one does in theory of disordered metals. A convenient way of calculations is to represent the solutions of equations (44) in a form that would allow one to integrate over the fields $\varphi_\mathbf{n}$ at the beginning of all the calculations. Integration over supervectors [37, 38] is most convenient for this purpose and we now follow this method. As the contribution of the density fluctuations is simple, we consider from now on the spin excitations only. The contribution of the density excitations will be added in the final results.

Using supervectors ψ and formulae for Gaussian integration we represent the partition function $\mathcal{Z}_\mathbf{h}$, equation (43), as follows [31]:

$$\mathcal{Z}_\mathbf{h} = \int \exp\left[2\nu\sqrt{2\mathrm{i}}\int \overline{\psi(X)}\mathcal{F}(X)\,\mathrm{d}X\right]\exp[-\mathcal{S}_\mathbf{h}[\psi]]D\psi \tag{50}$$

where

$$\mathcal{S}_\mathbf{h}[\psi] = -2\mathrm{i}\nu\int \overline{\psi(X)}\mathcal{H}\psi(X)\,\mathrm{d}X, \qquad \mathcal{H} = \mathcal{H}_0 - 2\mathrm{i}u\hat{h}(X)\tau_3. \tag{51}$$

The weight denominator in the Gaussian integral, equation (50), is absent because the ψ are supervectors. Although the general form of equations (50) and (51) is simple (it is a Gaussian integral), the detailed structure of the vectors and matrices is not. The supervectors ψ have 48 components. The number of components comes from the necessity to consider (1) three spin components (s-space), (2) bosonic and fermionic variables (g-space), (3) 'particles' and 'holes' (eh-space), (4) 'left' and 'right' motion (n-space), (5) one should double the number of the components to 'Hermitize' the space (H-space). The operator $\hat{L}_u(\mathbf{n})$, equation (45), is not Hermitian and the 'Hermitization' of the space of the supervectors is necessary to provide convergence of the Gaussian integral in equation (50).

The supervectors ψ are assumed to satisfy the bosonic periodicity conditions

$$\psi(\tau) = \psi(\tau + 1/T). \tag{52}$$

We emphasize that equation (52) holds for both the bosonic and fermionic components of the supervectors ψ.

The generalized coordinate X contains the components

$$X = (x, z), \qquad z = (\mathbf{n}, u) \tag{53}$$

where $\mathbf{n}$ is now a unit vector, $\mathbf{n}^2 = 1$, parallel to the original vector $\mathbf{n}$ but having only positive x components $n_x > 0$. Negative x-components are taken into account by doubling the number of components of the supervector ψ (n-space).

The operator $\mathcal{H}_0$ in equation (51) can be written as

$$\mathcal{H}_0 = -\mathrm{i}v_\mathrm{F}(\mathbf{n}\nabla)\tau_3\Sigma_3 - \Lambda_1\frac{\partial}{\partial\tau} \tag{54}$$

where Λ_1 is the first Pauli matrix in the Hermitized space (it is unity in all the other spaces) and the matrices τ_3 and Σ_3 are the third Pauli matrices in the eh and n spaces respectively

$$\Lambda_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_H, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{eh}, \qquad \Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_n. \tag{55}$$

The matrix $\hat{h}(X)$ has the spin structure of equation (46) (the replacement $\mathbf{n} \to \mathbf{n}\Sigma_3$ should be made) and is unity in all other spaces. The action $\mathcal{S}_\mathbf{h}[\psi]$ is supersymmetric and is invariant under homogeneous rotations in the superspace.

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The supervector $\bar{\psi}$ in equation (50) is conjugated with respect to ψ (see for the definition [31]). The vector $\mathcal{F}$ has the form

$$\mathcal{F}_{\mathbf{h}}(X) = \mathcal{F}_{\mathbf{h}}^1(X) + \mathcal{F}_{\mathbf{h}}^2(X) \tag{56}$$

where

$$\mathcal{F}_{\mathbf{h}\gamma}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_g \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}_H \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\text{eh}} \otimes \begin{pmatrix} h_\gamma(\mathbf{n}) \\ h_\gamma(-\mathbf{n}) \end{pmatrix}_n$$

$$\mathcal{F}_{\mathbf{h}\gamma}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}_g \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}_H \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\text{eh}} \otimes \begin{pmatrix} u\partial_\tau h_\gamma(\mathbf{n}) \\ u\partial_\tau h_\gamma(-\mathbf{n}) \end{pmatrix}_n$$

and $\gamma = x, y, z$ stands for the spin indices.

We see ((50), (51), (56)) that the Hubbard–Stratonovich field $h(X)$ enters both $\mathcal{H}$ and $\mathcal{F}$ linearly. We write the contribution of the spin excitations $\mathcal{Z}_t$ to the partition function as

$$\mathcal{Z}_t = \int \mathcal{Z}_{\mathbf{h}} W_t[\mathbf{h}] \mathcal{D}\mathbf{h}, \tag{57}$$

(cf equation (19)), which enables us to integrate immediately over $\mathbf{h}$.

The integration over $\mathbf{h}$ is Gaussian and can easily be performed. As a result, we obtain an effective action $\mathcal{S}$ containing not only the free quadratic part but also cubic and quartic interactions

$$\mathcal{Z}_t = \int \mathcal{D}\psi \exp(-\mathcal{S}), \qquad \mathcal{S} = \mathcal{S}_0 + \sum_{a=1,2,3} \mathcal{S}_a. \tag{58}$$

The interaction-independent part $\mathcal{S}_0$ equals

$$\mathcal{S}_0 = -2\mathrm{i}\nu \int \mathrm{d}X \, \overline{\psi_\alpha}(X) \mathcal{H}_0 \psi_\alpha(X). \tag{59}$$

The three different interaction terms present in the theory can be written as

$$\mathcal{S}_2 = -\mathrm{i}\nu \sum_{i,j=1}^4 \lambda_{ij} \int \mathrm{d}X \, \mathrm{d}X_1 (\overline{\psi}_\delta(X)\tau_3\Pi_j\partial_X\mathcal{F}_0)\Gamma_{X,X_1}^i (\overline{\mathcal{F}_0\partial_{X_1}}\Pi_j\tau_3\psi_\delta(X_1)) \tag{60}$$

$$\mathcal{S}_3 = -2\sqrt{-2}\mathrm{i}\nu \sum_{i,j=1}^4 \lambda_{ij}\varepsilon_{\delta\beta\gamma} \int \mathrm{d}X \, \mathrm{d}X_1 (\overline{\psi}_\delta(X)u\tau_3\Pi_j\psi_\beta(X))\Gamma_{X,X_1}^i (\overline{\mathcal{F}_0\partial_{X_1}}\tau_3\psi_\gamma(X_1)) \tag{61}$$

$$\mathcal{S}_4 = -2\nu \sum_{i,j=1}^4 \lambda_{ij}\varepsilon_{\delta\beta\gamma}\varepsilon_{\delta\beta_1\gamma_1}$$

$$\times \int \mathrm{d}X \, \mathrm{d}X_1 (\overline{\psi}_\beta(X)u\tau_3\Pi_j\psi_\gamma(X))\Gamma_{X,X_1}^i (\overline{\psi}_{\beta_1}(X)u_1\tau_3\Pi_j\psi_{\gamma_1}(X_1)). \tag{62}$$

Summation over spin indices $\alpha, \beta, \delta, \gamma$ is implied and we use the absolutely antisymmetric tensor $\varepsilon_{\alpha\beta\gamma}$ with $\varepsilon_{123} = 1$. The action $\mathcal{S}$, equations (58)–(62), is sufficient for calculation of the thermodynamic potential in the absence of a magnetic field. The operator $\Gamma_{X,X'}^i$ has the form

$$\Gamma_{X,X'}^i = \gamma_i(\widehat{\mathbf{n}\mathbf{n}'})f(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau') \tag{63}$$

and

$$\gamma_1(\widehat{\mathbf{n}\mathbf{n}_1}) = \left(\frac{\nu\hat{V}_t}{1 - 2\nu\hat{V}_t}\right)(\widehat{\mathbf{n}, \mathbf{n}_1}) \equiv \gamma_{\mathrm{f}} \tag{64}$$

$$\gamma_2(\widehat{\mathbf{n}\mathbf{n}_1}) = \left(\frac{\nu\hat{V}_t}{1 - 2\nu\hat{V}_t}\right)(\widehat{\mathbf{n}, -\mathbf{n}_1}) \equiv \gamma_{\mathrm{b}} \tag{65}$$

where $f(\mathbf{r})$ is the cut-off function introduced in equations (8) and (9). The operator ∂_X in equations (60)–(62) has the form

$$\partial_X(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & u\partial_\tau \end{pmatrix}_{\text{eh}}. \tag{66}$$

As we will see, the most interesting contribution comes from $\mathbf{n}$ and $\mathbf{n}_1$ nearly parallel to each other. This justifies the notations γ_{f} and γ_{b} standing for the bare forward and backward scattering.

The matrices Π_i equal

$$\Pi_1 = 1, \qquad \Pi_2 = \Sigma_3, \qquad \Pi_3 = \Lambda_1\tau_3, \qquad \Pi_4 = \Lambda_1\tau_3\Sigma_3 \tag{67}$$

and

$$\lambda_{ij} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{68}$$

The vector $\mathcal{F}_0$ in equations (60)–(62) has the form

$$\mathcal{F}_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_g \begin{pmatrix} 1 \\ 1_n \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{\text{eh}} \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix}_{\text{eh}} \end{pmatrix}_H. \tag{69}$$

The vector $\mathcal{F}_0$ projects on the bosonic sector and its presence violates the supersymmetry of the terms $\mathcal{S}_a, a = 2, 3$.

The action $\mathcal{S}$, (59)–(62) has the cubic and quartic interaction terms and looks quite complicated. Nevertheless, explicit calculations are not very difficult because perturbation theory in the interaction is logarithmic (see equations (48) and (49)) in all dimensions $d = 1, 2, 3$. In order to sum up the logarithms we use a renormalization group scheme.

At the end of this subsection we write additional terms in the action arising due to an external magnetic field $\mathbf{b}$ acting on spins. Writing these terms is necessary for calculation of the spin susceptibility. These terms can be brought to the form [32]

$$\mathcal{S}_{\text{b0}} = -\nu\eta \int dx\, \mathbf{b}^2(x) \tag{70}$$

$$\mathcal{S}_{\text{b1}} = -2\nu\sqrt{-2i}\eta \int dX\, b_\delta(x)(\overline{\psi}_{X,\delta}\tau_3\partial_X\mathcal{F}_0) \tag{71}$$

$$\mathcal{S}_{\text{b2}} = 4\nu\varepsilon_{\delta\beta\gamma}\eta \int dX\, b_\delta(x)(\overline{\psi}_{X,\beta}u\tau_3\psi_{X,\gamma}). \tag{72}$$

In these expressions

$$\eta = \frac{1}{1 - 2\nu\overline{V_t}}, \tag{73}$$

where the bar in $\overline{V_t}$ means averaging over the full solid angle.

3.3. Renormalization group equations and their solutions

We use a standard momentum shell renormalization group scheme. Separating fast and slow fields in the action we integrate over the fast fields and determine in this way the flow of coupling constants as a function of a running cut-off. In our case this amounts to a re-summation of the perturbation theory in the leading logarithmic approximation. We assume during the renormalization that the coupling constants γ are small, $\gamma \ll 1$.

In our case it convenient to define fast Φ and slow Ψ fields with respect to the frequency only. The reason is the anisotropy in momentum. As one can see, relevant momenta $p_\parallel$ are of the order of ω/v_F, while momenta $\mathbf{p}_\perp$ do not contribute to the logarithm and enter as parameters. Thus, we write

$$\psi(X) = \Psi(X) + \Phi(X), \tag{74}$$

where the fast fields Φ have the frequencies ω in the interval,

$$\kappa\omega_c < |\omega| < \omega_c$$

while the slow ones Ψ carry frequencies

$$|\omega| < \kappa\omega_c,$$

where ω_c is the running cut-off and $\kappa < 1$. Fast modes are integrated over in the Gaussian approximation using averages of the form

$$\langle\cdots\rangle_0 = \int d\Phi(\cdots)\exp(-\mathcal{S}_0[\Phi]). \tag{75}$$

This results in a change in $\mathcal{S}$

$$\delta\mathcal{S}[\Psi] = -\ln\langle\exp(-\mathcal{S}[\Psi + \Phi])\rangle_0 - \mathcal{S}[\Psi], \tag{76}$$

that will now be determined explicitly.

The main object of the RG calculations that we start now is the Green function $\mathcal{G}_0(\mathbf{k}, \mathbf{n}, \omega)$ corresponding to the bare action $\mathcal{S}_0$, equation (59). Calculating a Gaussian integral we obtain easily the form of the Green function in the Fourier representation

$$\mathcal{G}_0(\mathbf{k}, \mathbf{n}, \omega) = -4i\nu\langle\psi\bar\psi\rangle_0 = \frac{1}{i\omega\Lambda_1 + v_F\mathbf{k}\mathbf{n}\tau_3\Sigma_3} \tag{77}$$

where $\langle\cdots\rangle_0$ stands for averaging with $\mathcal{S}_0$, equation (59).

The bare Green function, equation (77), has a non-trivial matrix structure. Therefore renormalized vertices are also complicated and the form of the quadratic $\mathcal{S}_2$ and cubic terms $\mathcal{S}_3$ in the action, equations (60) and (61), is not most general. At the same time, the supersymmetric terms $\mathcal{S}_0$ and $\mathcal{S}_4$ do not change their form under the renormalization. Actually, $\mathcal{S}_0$ does not change in the first order and therefore we write the corresponding equations first for the effective vertices $\gamma_i(\xi)$, $i = 1, 2, 3, 4$, of the quartic interaction, where ξ is the running logarithmic variable.

A detailed calculation has been performed in [31]. It turns out that only the vertices γ_1 and γ_3 are non-trivially renormalized, whereas γ_2 and γ_4 remain equal to their bare values $\gamma_2^0 = \gamma_f$ and $\gamma_4^0 = \gamma_b$. At the same time, only $\gamma_3(\xi)$ enters the thermodynamic potential and we write here the RG equation for $\gamma_3(\xi)$ only

$$\frac{d\gamma_3(\xi)}{d\xi} = -[\gamma_3(\xi)]^2. \tag{78}$$

The renormalization of the $\mathcal{S}_2$ and $\mathcal{S}_3$ terms is more complicated. In order to obtain a closed system of RG equations one should write a more general form of the quadratic and cubic interactions. This is because the renormalization of different elements in the particle–hole space runs in a different way. As a result [31], there are two different cubic vertices β_i^+ and β_i^- and three quadratic vertices Δ_i^{++}, Δ_i^{--} and $\Delta_i^{+-} = \Delta_i^{-+}$ for all $i = 1, 2, 3, 4$. The bare values of all these vertices can be found from equations (60), (61) and they are equal to γ_f for $i = 1, 2$ and γ_b for $i = 3, 4$.

The vertices β_i and Δ_i do not renormalize for $i = 2, 4$ and remain equal to their bare values. Moreover, the vertices Δ_i do not renormalize for $i = 1$, too. The thermodynamic

potential is determined solely by $\Delta_3(\xi)$ and we write down the RG equations for $i = 3$ only. We remind the reader that this value of i corresponds to the backward scattering.

The equations for the cubic vertices $\beta_3^\pm(\xi)$ can be written as

$$\frac{d\beta_3^+(\xi)}{d\xi} = -2\gamma_3(\xi)\beta_3^+(\xi); \qquad \frac{d\beta_3^-(\xi)}{d\xi} = -\gamma_3(\xi)\beta_3^-(\xi). \tag{79}$$

The equation for $\Delta_3^{++}(\xi)$ takes the form

$$\frac{d\Delta_3^{++}(\xi)}{d\xi} = -2\Delta_3^{++}(\xi)\gamma_3(\xi) - 2[\beta_3^+(\xi)]^2; \tag{80}$$

whereas the equation for $\Delta_3^\pm(\xi)$ is

$$\frac{d\Delta_3^{-+}(\xi)}{d\xi} = \frac{d\Delta_3^{+-}(\xi)}{d\xi} = -2\beta_3^-(\xi)\beta_3^+(\xi). \tag{81}$$

The equation for $\Delta_3^{--}(\xi)$ can be written from the condition of the absence of ultraviolet divergences, which follows from the invariance of the system under spin rotations. It takes the form

$$\Delta_3^{--}(\xi)\gamma_3(\xi) = [\beta_3^-(\xi)]^2. \tag{82}$$

Equations (78)–(82) can easily be solved and their solutions satisfying the boundary conditions at $\xi = 0$ (when the vertices are equal to their bare values) take the form

$$\gamma_3(\xi) = \beta_3^-(\xi) = \Delta_3^{--}(\xi) = \frac{1}{\xi_b^* + \xi}; \tag{83}$$

$$\beta_3^+(\xi) = \Delta_3^{+-}(\xi) = \Delta_3^{-+}(\xi) = \frac{\xi_b^*}{(\xi_b^* + \xi)^2}; \tag{84}$$

$$\Delta_3^{++}(\xi) = \frac{2\xi_b^{*2}}{(\xi_b^* + \xi)^3} - \frac{\xi_b^*}{(\xi_b^* + \xi)^2}, \tag{85}$$

where we introduced the notation

$$\xi_b^* \equiv \frac{1}{\gamma_b} > 0$$

with the backscattering amplitude γ_b defined in equation (65). In equations (83)–(85) one should write the final value of the ξ at which the renormalization stops. Its value can be written as

$$\xi(\theta; u, u_1; \mathbf{r}_\perp) = u_1 u \mu_d \bar{f}(\mathbf{r}_\perp) \ln\left[\min\left(\frac{1}{\theta}, \frac{v_F}{r_0 T} \right) \right] \tag{86}$$

where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{n}_1$ and the function $\bar{f}$ is the Fourier transform of the function $f(k_\parallel = 0, \mathbf{k}_\perp)$, equation (9), with respect to $\mathbf{k}_\perp$. The parameter μ_d is given by $\mu_1 = 2$; $\mu_2 = 4(p_F r_0)^{-1}$; $\mu_3 = 4\pi(p_F r_0)^{-2}$.

One can see from equations (83)–(85) that all the relevant vertices decay as ξ grows. This is usually referred to as the 'zero-charge' situation. If we start from small values of the vertices (weak coupling) they become even smaller in the process of the renormalization and the one-loop approximation used here is sufficient for writing the final results.

4. Thermodynamic quantities

4.1. Specific heat

Calculation of thermodynamic quantities can be performed starting with a standard relation for the thermodynamic potential $\Omega(T)$

$$\Omega(T) = -T \ln \mathcal{Z} \tag{87}$$

where $\mathcal{Z}$ is the partition function.

We have performed the renormalization group calculations for the case when the vectors **n** and **n**$'$ of two spin excitations were close to each other (parallel or antiparallel motion). Only in this limit does one obtain large logarithms that determine the renormalization of the vertices. A crucial question is whether or not this narrow region of the parameters can bring an important contribution to thermodynamic or other physical quantities. This is not a trivial question because the system was not assumed to be one-or quasi-one-dimensional and one could imagine that all the effect of the singularities in the vertices would be washed out after the summation over the whole phase space.

In fact, this almost parallel motion of the spin excitations does not contribute much to the thermodynamic potential $\Omega(T)$ itself. Fortunately, this is not a very interesting quantity and what one would like to know are derivatives of the thermodynamic potential with respect to temperature and other parameters. One of the most important thermodynamic quantities is the specific heat C that can be expressed through the thermodynamic potential Ω as

$$C = -T \frac{\partial^2 \Omega}{\partial T^2}. \tag{88}$$

What we need is to calculate not the thermodynamic potential $\Omega(T)$ itself but the difference $\delta\Omega(T)$,

$$\delta\Omega(T) = \Omega(T) - \Omega(0). \tag{89}$$

Using the diagrammatic method of the calculations we should be able to express the thermodynamic potential $\Omega(T)$ in terms of sums of products of the Green functions over bosonic Matsubara frequencies ω_n

$$\Omega(T) = \sum_{\omega_n} R(\omega_n) \tag{90}$$

where $R(\omega_n)$ is a function of the frequency.

The sums of the type of equation (90) are very often divergent at high frequencies. This problem can be avoided calculating the quantity $\delta\Omega(T)$, equation (89).

Using the Poisson formula we represent $\delta\Omega(T)$ in the form

$$\delta\Omega(T) = 2 \sum_{l=1}^{\infty} \int \frac{\mathrm{d}\omega}{2\pi} R(\omega) \exp\left(-\frac{\mathrm{i}l\omega}{T}\right) \tag{91}$$

which improves drastically the convergence. The essential frequencies in equation (91) are of the order of T and are smaller than those frequencies that form logarithms in the vertices.

We calculate the partition function $\mathcal{Z}_t$ using equation (58). If we kept in the action $\mathcal{S}[\psi]$ the supersymmetric part $\mathcal{S}_0[\psi] + \mathcal{S}_4[\psi]$, equations (59) and (62), only, we would get zero for the thermodynamic potential Ω. The terms $\mathcal{S}_2[\psi]$ and $\mathcal{S}_3[\psi]$, equations (60) and (61), violate the supersymmetry and, as a result, one obtains finite contributions to Ω.

The thermodynamic potential $\Omega(T)$ can be expanded in terms of the renormalized action $\mathcal{S}_2[\psi]$ and the lowest orders of the expansion take the form

$$\Omega(T) = \Omega_1(T) + \Omega_2(T), \tag{92}$$

where

$$\Omega_1(T) = T\langle \mathcal{S}_2[\psi]\rangle_0, \qquad \Omega_2(T) = -\frac{T}{2}\langle(\mathcal{S}_2[\psi])^2\rangle_0 \tag{93}$$

and $\langle\cdots\rangle_0$ means averaging over ψ with the Lagrangian $\mathcal{S}_0[\psi]$, equation (59). The quantities $\delta\Omega_{1,2}(T)$ are obtained for $\Omega_{1,2}(T)$ by subtracting $\Omega_{1,2}(0)$.

It turns out that the terms $\delta\Omega_1(T)$ and $\delta\Omega_2(T)$ lead to qualitatively different types of contributions. In the first order in the interaction, only the forward scattering contributes. As we have found in the previous subsection, the part of $\mathcal{S}_2$ corresponding to the forward scattering is not renormalized and one may use just equation (60) for it. Then, we come to the expression

$$\frac{\delta C_1}{T} = \frac{\pi(3\gamma_f)}{6v_F\lambda_0^{d-1}} \tag{94}$$

where λ_0 differs from r_0, equation (9), by a numerical coefficient. Actually, λ_0 should be of the order of the wavelength if one wants to make estimates for the initial model of the interacting fermions. The factor 3 in the numerator is due to the fact that the spin-1 excitations have three projections. A similar contribution comes also from the density excitations but, of course, with the factor 1. Equation (94) describes a contribution of the interaction to the coefficient in front of the linear dependence of the specific heat on temperature.

The part $\delta\Omega_2$ consists of the part further renormalizing the coefficient C/T of the linear dependence on T and a part giving corrections non-analytic in T^2. A general expression for $\delta\Omega_2(T)$ can be brought to the form

$$\delta\Omega_2(T) = -6 \lim_{\eta\to+0} \sum_{l=1}^{\infty} \int \frac{d\omega}{(2\pi)} \exp\left(-i\frac{l\omega}{T} - \eta|\omega|\right)$$
$$\times \int dn_1\, dn_2 \int \frac{d^d k}{(2\pi)^d} Y(\theta; k_\perp, k_\parallel)\mathcal{P}_d(\omega, k; n_1, n_2) \tag{95}$$

where θ is the angle between the vectors n_1 and n_2. The main interesting contribution will come from small θ, which justifies the decomposition of the momentum k into perpendicular $k_\perp$ and parallel $k_\parallel$ with respect to $n_{1,2}$ components.

The function $Y(\theta; k_\perp, k_\parallel)$ defined as

$$Y(\theta; k_\perp, k_\parallel) = \int_0^1 \int_0^1 u_1 u_2\, du_1\, du_2\{[\Delta_3^{+-}(\theta; u_1, u_2; k_\perp, k_\parallel)]^2$$
$$+ \Delta_3^{++}(\theta; u_1, u_2; k_\perp, k_\parallel)\Delta_3^{--}(\theta; u_1, u_2; k_\perp, k_\parallel)\} \tag{96}$$

is the most important entry in the final expression for the specific heat. The vertices Δ_3 should be taken from equations (84)–(86).

The form factor

$$\mathcal{P}_d(\omega, n_1, n_2) = \frac{(i\omega + v_F n_2)(i\omega - v_F n_1)}{(i\omega - v_F n_2)(i\omega + v_F n_1)} \tag{97}$$

depends on the dimensionality of the system and it describes basically the free propagation of the two spin excitations in almost opposite directions. The non-analytic contributions originate from the small region of the phase space $|n_1 - n_2| \ll 1$.

The result of the calculation depends on the dimensionality of the system but the non-trivial corrections exist in both $d = 2$ and 3. The details of the calculations can be found in [31] but the final results for the non-analytic corrections δC to the specific heat look for $d = 2$ and 3 like

$$\delta C_{d=2} = -\frac{3\zeta(3)T^2}{\pi\varepsilon_F^2}\left\{[\gamma_b^\rho]^2 + \frac{12\gamma_b^2 \ln^2[1 + \mathcal{X}(T)/2]}{\mathcal{X}^2(T)}\right\} \tag{98}$$

$$\delta C_{d=3} = -\frac{3\pi^4}{10}\left(\frac{T}{\varepsilon_F}\right)^3 \left\{[\gamma_b^{\rho}]^2 \ln\frac{\varepsilon_F}{T} + \frac{6\pi\gamma_b}{\mu_3}\int_0^{\frac{\mathcal{X}(T)}{2\pi}}\frac{\mathrm{d}z}{z^2}[\mathrm{Li}_2(-z)]^2\right\} \tag{99}$$

where $\mathcal{X}(T) = \mu_d\gamma_b\ln(\varepsilon_0/T)$ ($\varepsilon_0 = v_F/r_0 \simeq \varepsilon_F$) and $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2$ is the polylogarithm function. The first terms in the circular brackets in equations (98) and (99) describe the density excitations and γ_b^{ρ} is the coupling constant for the backward scattering of these excitations. The second terms originate from the spin excitations. The final result, equations (98) and (99), was written for a special choice of the cut-off function $\bar{f}(\mathbf{r}_\perp)$, equation (86),

$$\bar{f}(\mathbf{r}_\perp) = \Omega_d^{-1}\exp(-r_\perp/r_0) \tag{100}$$

where Ω_d is the $(d-1)$-dimensional solid angle ($\Omega_1 = 2$, $\Omega_2 = 2\pi$).

In the limit of not very low temperatures when $\mathcal{X}(T) \ll 1$, equations (98) and (99) take a simpler form

$$\delta C_{d=2} = -\frac{3\zeta(3)}{\pi}\left(\frac{T}{\varepsilon_F}\right)^2([\gamma_b^{\rho}]^2 + 3\gamma_b^2) \tag{101}$$

$$\delta C_{d=3} = -\frac{3\pi^4}{10}\left(\frac{T}{\varepsilon_F}\right)^3\ln\left(\frac{\varepsilon_F}{T}\right)([\gamma_b^{\rho}]^2 + 3\gamma_b^2). \tag{102}$$

Equations (101) and (102) agree with results obtained by a direct diagrammatic expansion (see for a recent discussion [17, 18]). At the same time, the general equations (92) and (93) have been obtained for the first time in [31] using the bosonization scheme. We see that in the limit $T \to 0$ the contribution of the spin excitations vanishes and only the density excitations contribute to the specific heat.

The bosonization method presented here gives a possibility to make calculations also for one-dimensional systems. Surprisingly, the calculation of the 1D specific heat is more complicated than for $d = 2, 3$. This is because the function $\mathcal{P}_d(\omega, \mathbf{n}_1, \mathbf{n}_2)$, equation (97), is exactly equal to unity and the contribution of equation (95) to the specific heat vanishes. In order to obtain a non-vanishing contribution one should consider terms of higher orders in the effective vertices. As a result, one comes to the following expression for the correction to the specific heat in $d = 1$:

$$\delta C_{d=1} = \frac{\pi T}{v_F}\frac{1}{(1 + 2\gamma_b\ln\frac{\varepsilon_F}{T})^3}. \tag{103}$$

The correction to the specific heat for 1D can be extracted from the exact solution for spin chains of [39]. This correction agrees with our result, equation (103), and we see that our supersymmetric low energy theory reproduces all the previously known physical effects despite the fact that the intermediate degrees of freedom differ from the conventional bosonization.

4.2. Spin susceptibility

Calculation of the spin susceptibility has been performed in a recent work [32]. The bosonization scheme of [31] was used and it was shown that the logarithmic contributions arise as well. An external magnetic field **b** leads to the additional terms $\mathcal{S}_{b0}$, $\mathcal{S}_{b1}$ and $\mathcal{S}_{b2}$, equations (70)–(72), in the effective action.

We proceed as before integrating over the fast variables Φ, equation (74) and thus deriving renormalization group equations. In principle, one should renormalize not only the terms $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, equations (60)–(62), as was done previously, but also the terms $\mathcal{S}_{b0}$, $\mathcal{S}_{b1}$ and $\mathcal{S}_{b2}$, equations (70)–(72) and $\mathcal{S}_0$, equation (59). Again, the calculations should be done separately for $d = 1$ and $d = 2, 3$.

It turns out that in the higher dimensions $d = 2, 3$ the terms $\mathcal{S}_{b0}$, $\mathcal{S}_{b1}$ and $\mathcal{S}_{b2}$ and $\mathcal{S}_0$ do not change under the renormalization and what remains to be done is to express the susceptibility χ in terms of the renormalized vertices $\gamma(\xi)$, $\beta(\xi)$ and $\Delta(\xi)$. The computation is somewhat more cumbersome than the one for the specific heat and there are many contributions that have to be combined together. As for the specific heat, only the vertices with $i = 3$ contribute and one can express the susceptibility through $\gamma_3(\xi)$, $\beta_3(\xi)$ and $\Delta_3(\xi)$.

Using the solutions of the RG equations, equations (83)–(85), one can bring the expression for the susceptibility to a form containing the function $Y(\theta; \mathbf{k}_\perp, k_\parallel)$, equation (96). Choosing again the cut-off function $\bar{f}$ in the form of equation (100) we bring the temperature dependent correction $\delta\chi(T)$ to the spin susceptibility to the form [32]

$$\delta\chi_{d=2}(T) = 8\eta^2 \gamma_b^2 \frac{T}{\varepsilon_F} \chi_{d=2}^{(0)} \frac{\ln^2[1 + \mathcal{X}(T)/2]}{\mathcal{X}^2(T)} \tag{104}$$

$$\delta\chi_{d=3}(T) = \frac{(2\pi^2 \eta)^2}{3} \gamma_b^2 \left(\frac{T}{\varepsilon_F}\right)^2 \chi_{d=3}^{(0)} \frac{\{\mathrm{Li}_2[-\mathcal{X}(T)/2\pi]\}^2}{\mathcal{X}^2(T)} \tag{105}$$

where η is given by equation (73). The susceptibility $\chi_d^{(0)}$ is the Pauli susceptibility in d dimensions.

In the limit $\mathcal{X}(T) \ll 1$ one can neglect the logarithmic contributions and obtain the following form of the corrections $\delta\chi_d$:

$$\delta\chi_{d=2}(T) = 2\eta^2 \gamma_b^2 \frac{T}{\varepsilon_F} \chi_{d=2}^{(0)} \tag{106}$$

$$\delta\chi_{d=3}(T) = \frac{\pi^2}{3} \eta^2 \gamma_b^2 \left(\frac{T}{\varepsilon_F}\right)^2 \chi_{d=3}^{(0)}. \tag{107}$$

The linear dependence on T of the correction $\chi(T)$, equation (106), agrees with those obtained by conventional methods [13–15, 19]. At the same time, the proportionality of $\delta\chi(T)$ to T^2 in $d = 3$ is analytical in T^2 and equation (107) describes a renormalization of a coefficient in front of the T^2. As the T^2-term is present already in the temperature dependence of the susceptibility of the ideal Fermi gas, the correction is not very interesting. The first non-analytical term $T^2 \ln(\varepsilon_F/T)$ in the temperature dependence in $d = 3$ is proportional to γ_b^3.

Calculation of the susceptibility for one-dimensional systems is somewhat more involved because one should consider corrections to the terms $\mathcal{S}_{b0}$, $\mathcal{S}_{b1}$ and $\mathcal{S}_{b2}$, equations (70)–(72), and $\mathcal{S}_0$, equation (59). Nevertheless, one can proceed in a rather straightforward way and, as a result, the following dependence of the susceptibility is obtained [32]:

$$\delta\chi_{d=1}(T) = \frac{2\nu\gamma_b}{1 + 2\gamma_b \ln(\varepsilon_F/T)}. \tag{108}$$

This result agrees the with the one obtained long ago [40] by a completely different RG method developed for the electron problem in 1D. So, equation (108) serves as one more check of the bosonization method reviewed here.

5. Discussion

In the previous sections we reviewed the new method of bosonization for a clean Fermi gas in any dimensions suggested in [31] and further used in [32]. In contrast to previous attempts [21, 22, 6, 23–25, 27, 28, 5], we do not restrict ourselves to the case of a long range electron–electron interaction and include in the scheme spin degrees of freedom. This enables us to consider not only density excitations but also the spin ones. In contrast to the

density excitations that can be described by a scalar, the spin excitations are described by three-component vectors and one may speak about a new version of the non-Abelian bosonization.

The non-Abelian character of the effective theory leads to the non-trivial interaction between the spin modes. Making the perturbation theory in this interaction new logarithmic contributions (diverging in the limit $T \rightarrow 0$) to vertices were discovered and summed up using a RG scheme. As a result of this consideration we have found temperature dependent corrections $\delta C(T)/T$ to the specific heat and to the susceptibility $\delta \chi(T)$ in all dimensions, equations (98), (99), (103), (104), (105). The results for $d = 1$ agree with those obtained earlier by completely different methods. In higher dimensions, $d = 2, 3$ the lowest order of the expansion of our formulae in the coupling constant agrees with known results obtained previously using conventional diagrammatic techniques, see e.g. [17, 18, 13–15, 19]. All these agreements serve as a good check of our approach.

The new contributions to the specific heat and susceptibility originate from an almost parallel motion of the spin excitations. Although both the forward and backward scattering amplitudes are renormalized, only the backward scattering enters the thermodynamic quantities.

In the language of conventional diagrams the logarithmic contributions to the thermodynamic quantities come from both Cooper and particle–hole loops because they originate from quasi-one-dimensional processes. The fact that the forward scattering amplitude drops out from the final results corresponds to what happens in 1D.

Although we have agreement in the limiting cases with almost all the results we could compare with, there is a disagreement with an old work [41], where an instability of the Fermi liquid against superconducting pairing with a high angular momentum was found. This is especially strange because the main contribution to the formation of the superconductivity comes from almost parallel electron motion such that only forward and backward scattering enter the superconducting critical temperature. This is just the region that we considered. As we do not see any such effect, this can mean that either (1) the accuracy of our method is not sufficient (we summed the terms like $(\gamma \ln(\varepsilon_F/T))^n$, whereas $(\gamma^2 \ln(\varepsilon_F/T))^n$ were summed in [41]) or (2) summing the Cooper ladder as was done in [41] is a bad approximation for the quasi-one-dimensional process. This question deserves a more detailed study.

We have developed and applied the bosonization scheme for the simplest model of the interacting Fermi gas. One can add other terms to the Hamiltonian of this model to take into account different features relevant for experimental systems. The logarithmic contributions discussed in the previous sections should be very pronounced near quantum phase transitions into a magnetic state where the spin fluctuations may be very strong (for a recent review of such systems see, e.g., [42]). The anomalous contributions of the spin excitations can be important in doped Mott insulators (see e.g. [43]). A two-dimensional silicon metal–oxide–semiconductor field effect transistor (Si-MOSFET) [44, 45] is one more system where the effects discussed in the present paper can be important [46]. We hope that our scheme may be useful in study of these systems.

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